Deriving the imaginary unit and Euler's formula from first principles, and discovering the existence of rotation factor set

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Abstract

Angle-dependent rotation factors have been introduced to describe position vector rotation in a plane, and a fundamental equation of position vector rotation that governs the rule for the rotation of a point in a plane has been discovered. The imaginary unit turns out to be an orthogonal rotation factor that directly appears in the fundamental equation. This finally reveals the causality origin of the mysterious imaginary unit. Based on the fundamental equation, the existence of the rotation factor set for all rotation angles has been found and the formula for the set has been obtained. The rotation factor set can be used to construct coordinate systems. The result of constructing a Cartesian coordinate system by rotation factors is equivalent to the complex number system with the imaginary unit being the orthogonal rotation factor and Euler's formula being the consequence of the rotation factor set.

1. Introduction

The discovery of the imaginary unit i from solving the cubic equation formed the complex numbers that were eventually accepted by renaissance mathematicians [1, 2, 7, 8, 10] and have deep significance and profound importance to our understanding of mathematics and physics [12, 13].

From the cause-and-effect viewpoint, it is natural to ask the basic question: Is the imaginary unit the cause or the effect? If the imaginary unit is the effect, it is generally unfeasible to find the cause by studying the effect within the effect's scope or its extension developed based on the effect. However, the cause being unknown may not hinder the effect's development and its applications.

In 1748, Leonhard Euler obtained [3, 4] Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ without using the concept of the complex plane, which was not yet introduced. The equation was called "the most remarkable formula in mathematics" by the physicist Richard Feynman [5], and lies at the heart of complex number theory [11]. Euler's identity $e^{i\pi} = -1$, which is the result of Euler's formula with the angle set to π, is considered to be an exemplar of mathematical beauty [9, 11, 14, 19].

About 50 years later after the creation of Euler's formula, Caspar Wessel described complex numbers as points in the complex plane [17, 18]. With the geometric significance of the complex plane for complex analysis, the complex number system has captivated more than 150 years of intensive development, finding diverse applications in science and engineering [12, 13].

Over the course of the development of the complex number system, through geometric representation, the exploration of the nature of the imaginary unit has enriched the understanding that indicates the rotation consequences generated by the imaginary unit [11, 12].

Today, from the causality perspective, the cause of the imaginary unit remains unknown although the consequences it generates have been well studied.

2. Rotation factors

From this point on (until specifically mentioned), assume that there is no existence of the complex number system so that our mind is not influenced by the existing concepts and will focus on creating a number system for describing rotation from first principles.

First, from a high level perspective, one may expect the existence possibility of a new number system for describing rotation. Nature has its laws and properties. Mathematics is invented to discover and describe Nature. Real numbers represent points on a number line, and can describe stretching and displacement of a point's position by multiplication and addition. On the other hand, rotation of a point's position is also a fundamental motion type besides the stretching and displacement. From the viewpoint of Nature being perfect, there likely also exists a number system for describing the rotation.

A point in the one-dimension number line represented by real numbers cannot perform a rotation motion. The next higher dimension, a two-dimension plane is needed and the position of a point in the plane may be represented by a position vector. Real numbers can only make the vector stretch by multiplication.

By analogy with real numbers, rotation numbers are introduced to make the vector rotate by multiplication. Mathematics represents abstractions of properties of the physical world. Rotation numbers may also be called rotation factors that are more in line with causality and physics convention.

2.1. Terminologies and conventions

Rotation factor: The term rotation factor is mainly used here. It may also be called rotation number.

Position vector: A position vector represents the position of a point. The direction of a position vector is from the coordinate origin to the point represented by the vector, and the magnitude of the vector is the distance between the point and the origin.

Position number: Almost identical to a position vector in the representation sense, a position number represents the position of a point. The direction of a position number is from the coordinate origin to the point represented by the number, and the magnitude of the number is the distance between the point and the origin.

Position vector and position number conventions: Here the terms position vector and position number are interchangeable for representing a point. For geometric representation, vector arrow may be used for a position number to explicitly indicate the direction of the position number.

Point of position vector or position number: means the point represented by the position vector or the position number, and vice versa.

Direction of point, position vector, or position number: means the direction from the origin point to the point.

Rotating (or rotation of) point, position vector, or position number: means rotating a point to another position with respective position vector change or position number change.

Applying a rotation factor to: means multiplying a rotation factor to a position vector or a position number. Applying a rotation factor to a point means applying the factor to the point's position vector or position number.

Target of a rotation factor: means the position vector or the position number that a rotation factor is applied to.

2.2. Definition of rotation factors

An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position vector of a point and making the vector rotate counter-clockwise by angle δ relative to the origin point of the vector with the vector's magnitude unchanged. That is

$$\mathbf{P}_\delta = \mathbf{P} \cdot \mathrm{q} = \mathrm{q} \cdot \mathbf{P} \tag{1}$$

where (see Figure 1) Point P represented by position vector $\mathbf{P}$ rotates relative to origin point O by angle δ and reaches Point P_δ represented by position vector $\mathbf{P}_\delta$.

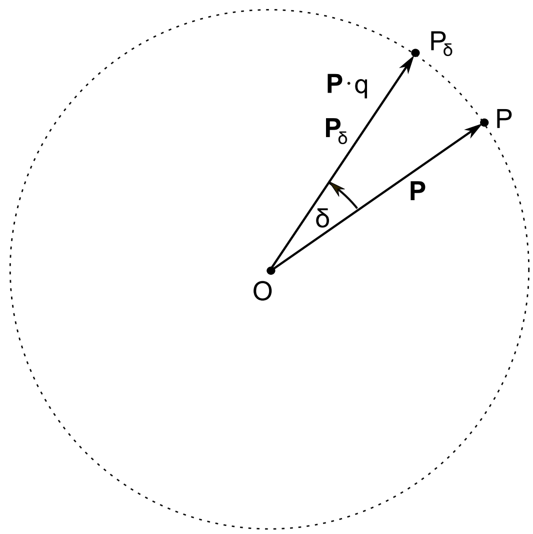

Figure 1. Rotation factor and target vector's direction change

Since the rotation factor does not change vector's magnitude, the relationship for respective unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}_\delta$ of $\mathbf{P}$ and $\mathbf{P}_\delta$ in (1) becomes

$$\hat{\mathbf{u}}_\delta = \hat{\mathbf{u}} \cdot \mathrm{q} \tag{2}$$

At this point, the existence of rotation factors such defined is a postulate. If the consequences of the postulate are consistent with existing results, it is then considered true.

2.3. Attributes of rotation factors

Denote a rotation factor q of an arbitrary angle δ as $\mathrm{q}(\delta)$. The following attributes of rotation factors are implied or derived from the definition (1).

Attribute 1. Being a new type of number with unit magnitude

A rotation factor is not a real number as the former can change a vector's direction by multiplication and the latter cannot. It may be considered as a new type of number. A rotation factor of an arbitrary angle always has a unit magnitude, and will not change the target vector's magnitude.

Attribute 2. Implying a direction relative to its target vector

It is important to note that a rotation factor of an angle has no direction itself. However, when being applied to a vector, it starts to imply a direction relative to the target vector. The rotation factor will make the target vector rotate to the implied direction.

Attribute 3. Having a period of 2π

After a 2π rotation, a vector's direction remains the same. That is

$$\mathrm{q}(\delta + n \cdot 2\pi) = \mathrm{q}(\delta)$$

where n is an integer.

Attribute 4. Negative angle representing clockwise rotation and canceling positive angle rotation effect

A rotation factor of angle δ produces a counter-clockwise rotation for a target vector, which may be cancelled by a rotation factor of negative angle $-\delta$ and vice versa. If a rotation factor produces no rotation, it is corresponding to 1. That is

$$\mathrm{q}(\delta) \cdot \mathrm{q}(-\delta) = 1$$

$$\mathrm{q}(-\delta) \cdot \mathrm{q}(\delta) = 1$$

Attribute 5. Division operation of rotation factor being allowed

The multiplication effect produced by a rotation factor may be inversed through division by the same rotation factor and vice versa. This means

$$\mathrm{q} \cdot \frac{1}{\mathrm{q}} = 1$$

$$\frac{1}{\mathrm{q}} \cdot \mathrm{q} = 1$$

Attribute 6. Rotation factors being commutative when being applied to a vector

The order of rotation factors for multiplying to a vector will not affect the final result. That is

$$\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2) = \mathrm{q}(\delta_2) \cdot \mathrm{q}(\delta_1)$$

Attribute 7. Multiplications to a vector by multiple rotation factors being equivalent to one rotation factor with the sum of the individual angles

If the total number of the multiple rotation factors is n. This attribute is expressed by

$$\mathrm{q}(\delta_1) \cdot \mathrm{q}(\delta_2) \cdot ... \cdot \mathrm{q}(\delta_n) = \mathrm{q}(\delta_1 + \delta_2 + ... + \delta_n)$$

Attribute 8. Becoming real numbers 1 and -1 respectively for angles 0 and π

In general, a rotation factor of an arbitrary angle is not a real number. But the two angles 0 and π are the exceptions. The rotation factor of angle 0 produces no rotation for its target vector and becomes the real number 1. And the rotation factor of angle π produces a reverse direction for its target vector by rotating 180° and becomes the real number -1. That is

$$\mathrm{q}(0) = 1$$

$$\mathrm{q}(\pi) = -1$$

Furthermore, assume that angle δ times integer n equals the full circle angle 2π ($\mathrm{n} \cdot \delta = 2\pi$; $\delta = \frac{2\pi}{n}$) and n is an even number. After n times of multiplication operations to a vector by the angle's rotation factor, the final effect is no rotation as the vector rotates by a full circle angle and comes back to the initial direction. This is equivalent to the final rotation factor being 1. Similarly, after $\frac{n}{2}$ times of the multiplication operations, the final effect is that the vector is rotated by π and the vector's direction is reversed. This is equivalent to the final rotation factor being -1. The two cases are respectively expressed by

$$\mathrm{q}(\frac{2\pi}{n})^n = 1 \tag{3}$$

$$\mathrm{q}(\frac{2\pi}{n})^{n/2} = -1 \tag{4}$$

3. Orthogonal rotation factor

Since the two angles 0 and π are exceptional and respectively corresponding to rotation factors being the real numbers 1 and -1, it is natural to think that the exact middle angle $\frac{\pi}{2}$ between 0 and π is also special.

The rotation factor of the angle $\frac{\pi}{2}$ is here defined as orthogonal rotation factor in that it produces a 90° rotation for the target vector with the resultant direction being perpendicular or orthogonal to the initial direction.

Notationwise, the orthogonal rotation factor is denoted as $\mathrm{q}(\frac{\pi}{2})$, which may be further succinctly denoted by a symbol i. That is

$$i = \mathrm{q}(\frac{\pi}{2}) \tag{5}$$

In Figure 2, Point P relative to origin Point O forms its position vector $\mathbf{P}$. Multiplying the orthogonal rotation factor i to vector $\mathbf{P}$ makes the vector rotate by angle $\pi/2$ or 90° with the resultant vector indicated by $\mathbf{P} \cdot i$ and the resultant point position indicated by Point $\mathrm{P}_{\pi/2}$.

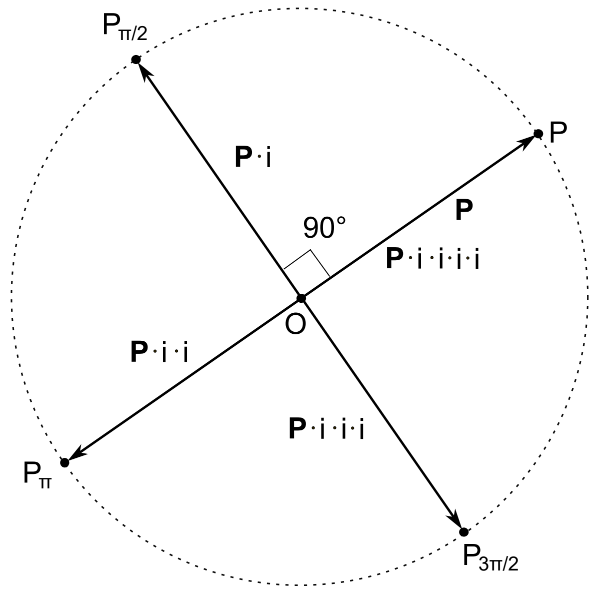

Figure 2. Orthogonal rotation factor being successively applied to position vector

Further, multiplying i to vector $\mathbf{P} \cdot i$ makes the vector rotate by 90° with the resultant vector indicated by $\mathbf{P} \cdot i \cdot i$ and the resultant point position indicated by Point P_π. It is noted that the resultant vector has a reversed direction relative to the initial position vector $\mathbf{P}$. That is, $\mathbf{P} \cdot i \cdot i = -\mathbf{P}$ or $i \cdot i = -1$ or $i^2 = -1$.

And further, multiplying i to vector $\mathbf{P} \cdot i \cdot i$ makes the vector rotate by 90° with the resultant vector indicated by $\mathbf{P} \cdot i \cdot i \cdot i$ and the resultant point position indicated by Point $P_{3\pi/2}$. The resultant vector has a reversed direction relative to the position vector $\mathbf{P} \cdot i$. That is, $\mathbf{P} \cdot i \cdot i \cdot i = -\mathbf{P} \cdot i$ or $i^3 = -i$.

And finally, multiplying i to vector $\mathbf{P} \cdot i \cdot i \cdot i$ makes the vector rotate by 90° with the resultant vector indicated by $\mathbf{P} \cdot i \cdot i \cdot i \cdot i$ and the resultant point position indicated by P. It is noted that the resultant vector is the same as the initial position vector $\mathbf{P}$. That is, $\mathbf{P} \cdot i \cdot i \cdot i \cdot i = \mathbf{P}$ or $i^4 = 1$.

It is important to note the case where multiplying the orthogonal rotation factor to vector $\mathbf{P}$ twice and makes the vector rotate by 180° with a resultant direction being reversed. That is, $\mathbf{P} \cdot i \cdot i =$ - $\mathbf{P}$ or

$$i^2 = -1 \tag{6}$$

This result (6) may also be obtained from formula (4). In (4), with n = 4 for the orthogonal rotation factor, the formula becomes $\mathrm{q}(\frac{\pi}{2})^2 = -1$. By definition (5), the formula then becomes (6).

The result (6) has its counterpart for clockwise orthogonal rotation, which is derived next. In Figure 2, it is apparent that $\mathbf{P} \cdot \mathrm{q}(-\frac{\pi}{2}) = -\mathbf{P} \cdot \mathrm{q}(\frac{\pi}{2})$. This means that vector $\mathbf{P}$

rotated clockwise by $\frac{\pi}{2}$ has the opposite direction to vector $\mathbf{P}$ rotated counter-clockwise by $\frac{\pi}{2}$. Therefore, $\mathrm{q}(-\frac{\pi}{2}) = -\mathrm{q}(\frac{\pi}{2})$ or $\mathrm{q}(-\frac{\pi}{2}) = -i$. This states that $-i$ is equivalent to the rotation factor that makes a vector rotate clockwise by $\frac{\pi}{2}$. Multiplying $(-i)$ twice to $\mathbf{P}$ makes the vector rotate by 180° clockwise with the resultant direction being reversed. That is, $\mathbf{P} \cdot (-i) \cdot (-i) = -\mathbf{P}$ or $(-i) \cdot (-i) = -1$, which leads to

$$(-i)^2 = -1 \tag{7}$$

From (6) and (7), it is apparent that equation

$$z^2 = -1 \tag{8}$$

has two solutions $\pm i$ where i is the orthogonal rotation factor. This also directly shows that the orthogonal rotation factor is not a real number as the square of a real number is always positive.

4. Fundamental equation of position vector rotation

Since a rotation factor is involved in a position vector's rotation, it is natural to study the rotation motion of the position vector.

In Figure 3, Point P is represented by position vector $\mathbf{P}$ with magnitude r. $\hat{\mathbf{n}}$ is the unit vector perpendicular to the position vector $\mathbf{P}$ at Point P. Point P rotates around origin Point O by angle $\Delta\theta$ with the vector's magnitude r unchanged, and reaches Point $\mathrm{P}_{\Delta\theta}$ whose position vector is $\mathbf{P}_{\Delta\theta}$. $\mathbf{\Delta P}$ is the vector change, which is the difference between vector $\mathbf{P}_{\Delta\theta}$ and vector $\mathbf{P}$.

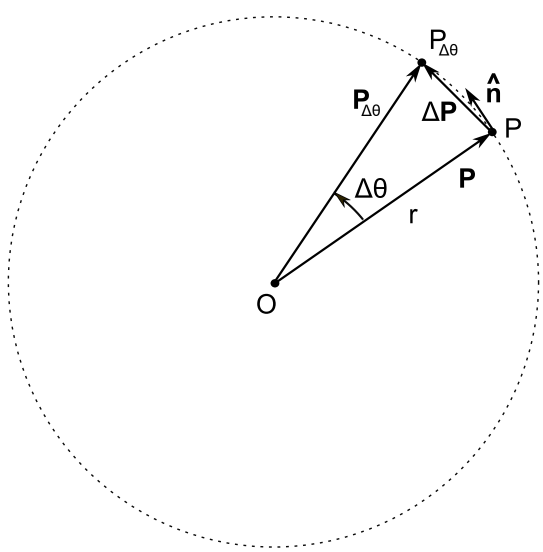

Figure 3. Position vector rotation relative to an origin point

Since the vector's magnitude r is unchanged, Point P moves along the circumference of the circle during the rotation. Let Δs be the length of the arc corresponding to the angle change $\Delta\theta$, and Δl be the length of the chord of the arc. The Δl is also the magnitude of the vector change $\mathbf{\Delta P}$. That is, $\Delta l = \|\mathbf{\Delta P}\|$. Denote $\hat{\mathbf{u}}$ as the unit vector of $\mathbf{\Delta P}$. Then

$$\mathbf{\Delta P} = \|\mathbf{\Delta P}\|\,\hat{\mathbf{u}} = \Delta l\hat{\mathbf{u}} \tag{9}$$

For a circle, the ratio of the arc length to the arc angle span ($\Delta\theta$) is always equal to the radius regardless of the value of the angle span. That is, $\frac{\Delta s}{\Delta\theta} = r$ or

$$\Delta\theta = \frac{\Delta s}{r} \tag{10}$$

When $\Delta\theta$ is small, the arc length Δs is approximately equal to the chord length Δl. That is, $\Delta l \cong \Delta s$ or

$$\frac{\Delta l}{\Delta s} \cong 1 \tag{11}$$

Also when $\Delta\theta$ is small, the direction of $\hat{\mathbf{u}}$ is close to that of $\hat{\mathbf{n}}$. That is

$$\hat{\mathbf{u}} \cong \hat{\mathbf{n}} \tag{12}$$

From (9) and (10), the ratio of the vector change to the angle change is expressed as

$$\frac{\mathbf{\Delta P}}{\Delta\theta} = (\frac{\Delta \mathrm{l}}{\Delta s})r\hat{\mathbf{u}} \tag{13}$$

As $\Delta\theta \to 0$, $(\frac{\Delta \mathrm{l}}{\Delta s}) \to 1$ and $\hat{\mathbf{u}} \to \hat{\mathbf{n}}$, the limit of (13) for the derivative with respect to θ leads to

$$\frac{\partial \mathbf{P}}{\partial\theta} = r\hat{\mathbf{n}} \tag{14}$$

It is noted that multiplying the orthogonal rotation factor i to the vector $\mathbf{P}$ makes the vector perform an orthogonal rotation with the vector's magnitude r unchanged. In other words, the resultant vector has the magnitude r and the direction of $\hat{\mathbf{n}}$. That is

$$i\mathbf{P} = r\hat{\mathbf{n}} \tag{15}$$

This makes (14) become

$$\frac{\partial \mathbf{P}}{\partial\theta} = i\mathbf{P} \tag{16}$$

The result (16) serves as a fundamental equation of position vector rotation. It governs the rule for the rotation of a point in a plane. The rule says that a point's position vector change rate with respect to rotation angle is equal to the point's position vector multiplied by the orthogonal rotation factor. The equation is surprisingly succinct and elegant. The orthogonal rotation factor i happens to appear in this fundamental equation.

5. Fundamental equation of position number rotation and discovery of rotation factor set

For representing a point, a position number and its corresponding position vector have the same direction and magnitude. The direction of a position number is implicit or implied in contrast to the position vector whose direction is explicit. In the definition (1), a rotation factor of an arbitrary angle is applied to a position vector, which has a direction. In fact, the same rotation factor can also be applied to a position number. The rotation factor will generate the same rotation for the position number regardless of the target's direction being explicit or implied.

The definition (1) is here restated for position number with an implied direction. An angle-dependent rotation factor q of an arbitrary angle δ is defined as a factor for multiplying to a position number p of a point and making the position number's implied direction rotate counter-clockwise by angle δ with the position number's magnitude unchanged. That is

$$\mathrm{p}_\delta = \mathrm{p} \cdot \mathrm{q} = \mathrm{q} \cdot \mathrm{p} \tag{17}$$

where p_δ is the resultant position number after the rotation.

Similar to Equation (16) for position vector, the fundamental equation of position number rotation is derived next.

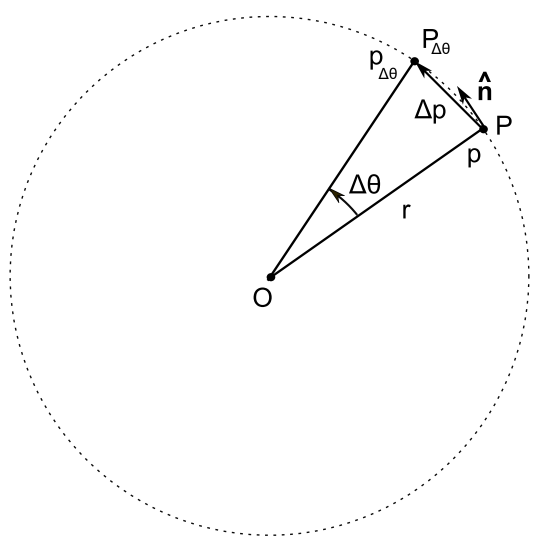

Figure 4. Position number rotation relative to an origin point

Figure 4 illustrates the use of a position number to represent the rotation of a point. Figure 4 is the same as Figure 3 except that position vectors in Figure 3 are replaced with corresponding position numbers in Figure 4. The process for obtaining the fundamental equation for the position number is similar to the position vector's. Since the two are not exactly the same, we still go through the process by re-using the same prior variable definitions and handling the differences accordingly.

The Δl is also the magnitude of the position number change $\Delta \mathrm{p}$. That is, $\Delta l = |\Delta \mathrm{p}|$. Denote $\hat{\mathbf{u}}$ as the unit vector of the implied direction of $\Delta \mathrm{p}$. Then

$$\Delta \mathrm{p} = |\Delta \mathrm{p}|\hat{\mathbf{u}} = \Delta l \hat{\mathbf{u}} \tag{18}$$

From (10) and (18), the ratio of the position number change to the angle change is expressed as

$$\frac{\Delta \mathrm{p}}{\Delta \theta} = (\frac{\Delta \mathrm{l}}{\Delta s}) r \hat{\mathbf{u}} \tag{19}$$

$\hat{\mathbf{n}}$ is the unit vector of the direction perpendicular to the implied direction of the position number p. As $\Delta\theta \to 0$, $(\frac{\Delta \mathrm{l}}{\Delta s}) \to 1$ and $\hat{\mathbf{u}} \to \hat{\mathbf{n}}$, the limit of (19) for the derivative with respect to θ leads to

$$\frac{\partial \mathrm{p}}{\partial \theta} = r\hat{\mathbf{n}} \tag{20}$$

It is noted that the implied direction of the position number p is perpendicular to $\hat{\mathbf{n}}$, and multiplying the orthogonal rotation factor i to the position number p makes the implied direction rotate to the $\hat{\mathbf{n}}$ direction with the position number's magnitude r unchanged. In other words, the resultant position number has the magnitude r and the direction of $\hat{\mathbf{n}}$. That is

$$i\mathrm{p} = r\hat{\mathbf{n}} \tag{21}$$

It follows from (20) and (21) that

$$\frac{\partial \mathrm{p}}{\partial \theta} = i\mathrm{p} \tag{22}$$

Equation (22) is the fundamental equation of position number rotation. To solve this equation, the dependence of the position number on the rotation angle θ is obtained from Figure 5 based on the definition of the rotation factor.

In Figure 5, on a plane, RE denotes a real number axis and OR denotes the axis that is orthogonal to the real number axis. Point O is the origin. Point $\mathrm{P_r}$ is a point in the real number axis, and is represented by real number r. The implied direction of Point $\mathrm{P_r}$ is the positive direction of the real number axis. $\mathrm{q}(\theta)$ is the rotation factor of angle θ. Multiplying the rotation factor $\mathrm{q} = \mathrm{q}(\theta)$ to the real number r makes its implied direction rotate by angle θ with magnitude r unchanged. As a result, Point $\mathrm{P_r}$ is rotated to Point P, which is represented by position number p. Before the rotation, the position number is r. After the rotation, the position number is $\mathrm{rq}(\theta)$. By definition, the resultant position number, which is p, equals $\mathrm{rq}(\theta)$. That is

$$\mathrm{p} = \mathrm{rq}(\theta) \tag{23}$$

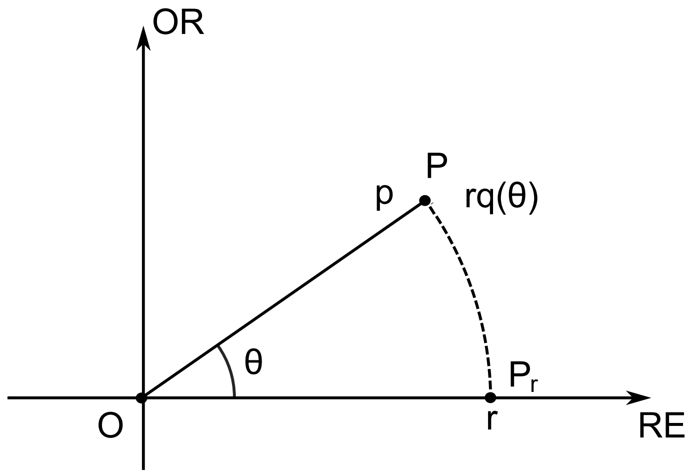

Figure 5. Position number represented by a real number multiplied by a rotation factor

With (23), the equation (22) becomes

$$\frac{\partial(r\mathrm{q})}{\partial\theta} = ir\mathrm{q} \tag{24}$$

where $\mathrm{q} = \mathrm{q}(\theta)$.

Since q depends on θ only and r is considered as constant with respect to change in θ, it follows that $\frac{\partial(r\mathrm{q})}{\partial\theta} = r\frac{\partial\mathrm{q}}{\partial\theta} = r\frac{\mathrm{dq}}{d\theta}$ and (24) leads to

$$\frac{\mathrm{dq}}{d\theta} = i\mathrm{q} \tag{25}$$

From the previous discussion, division is allowed for rotation factor including the orthogonal rotation factor i. Then (25) becomes

$$\frac{\mathrm{dq}}{d(i\theta)} = \mathrm{q} \tag{26}$$

Let $\phi = i\theta$. It follows from (26) that

$$\frac{\mathrm{dq}}{d\phi} = \mathrm{q} \tag{27}$$

The equation (27) says that the derivative of q with respect to ϕ equals q itself. Thus, q is the exponential function. That is, $\mathrm{q} = e^{\phi}$. Since $\phi = i\theta$, then $\mathrm{q} = e^{i\theta}$. With $\mathrm{q} = \mathrm{q}(\theta)$, it follows that

$$\mathrm{q}(\theta) = e^{i\theta} \tag{28}$$

The result (28) is a rotation factor formula for obtaining a rotation factor of a specific angle θ. The formula shows the existence of rotation factors covering all angles and has profound implications.

6. The existence of rotation factor set

It is important to note that although the rotation factor formula (28) is derived from a polar coordinate form, the angle of a rotation factor in the definition (1) or (17) as well as in (28) is relative to the target's direction and independent of coordinate systems. This relativeness property of a rotation factor is found to be powerful and useful.

Due to the relativeness and rotation properties, the rotation factors can be used to construct coordinate systems. For a polar coordinate system, from (23) and (28), the position number p is expressed as

$$\mathrm{p} = re^{i\theta} \tag{29}$$

This formula (see Figure 5) means that multiplying a rotation factor of angle θ to a real number r in a real number axis makes the real number point rotate to the point represented by the position number p. In other words, the position number is formed by multiplying a real number with a rotation factor.

All real numbers form a field [16]. The set of real numbers is denoted $\mathbb{R}$ [15]. In comparison, rotation factors are a new kind of number. The rotation factor formula (28) is an exponential function and exponential field. That is, all rotation factors of different angles represented by the formula form a field. The set of rotation factors is then given as

$$\mathbb{E} = \{e^{i\theta} \mid \theta \in \mathbb{R}, 0 \le \theta \le 2\pi\} \tag{30}$$

It is noted that althrough important and special, the orthogonal rotation factor i is just one member in the set E.

Let $\mathbb{P}$ denote the set of all position numbers. $\mathbb{P}$ represents all points in a plane. A member in $\mathbb{P}$ may be constructed by a member in $\mathbb{R}$ and a member in $\mathbb{E}$. Figure 6 illustrates their relationship.

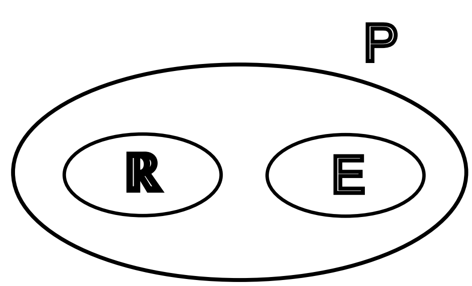

Figure 6. Real number set $\mathbb{R}$, rotation factor set $\mathbb{E}$ and position number set $\mathbb{P}$

The rotation factor set $\mathbb{E}$ may be represented by the unit circle in Figure 7. In the figure, RE denotes a real number axis and OR denotes the axis orthogonal to RE. For angle 0, the rotation factor $e^{i\cdot 0} = 1$, which means the point represented by unit number 1 is not rotated; For angle θ, the point of unit number 1 is rotated to the point represented by $e^{i\cdot\theta}$ after unit number 1 being multiplied by the rotation factor; In the same way, for angle $\frac{\pi}{2}$, the point of unit number 1 is rotated to the point represented by $e^{i\cdot\frac{\pi}{2}} = i$; For angle π, the point of unit number 1 is rotated to the point represented by $e^{i\cdot\pi} = -1$ with the position's direction reversed.

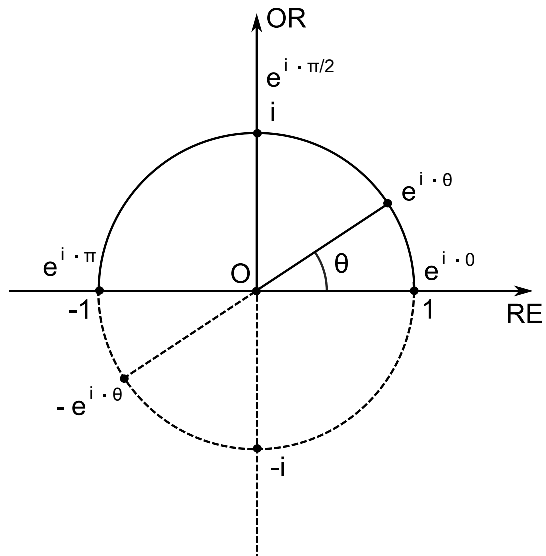

Figure 7. Unit circle representation for the rotation factor set

The upper-half of the unit circle is represented by the angle range of 0 to π. And the lower-half of the circle is represented by the range of π to 2π. Each rotation factor on the lower-half of the circle can be represented by a corresponding rotation factor on the upper-half of the circle with the direction reversed or multiplied by -1.

By analogy with positive and negative real numbers, the set of rotation factors represented by the upper-half of the circle (solid-line) may be called the positive set of rotation factors; and the set of rotation factors represented by the lower-half circle (dashed-line) may be called the negative set of rotation factors. The negative set can be represented by the positive set with its members multiplied by -1. The positive set and negative set form the full set $\mathbb{E}$.

It is noted that the angles forming the unit circle in Figure 7 are relative to the fixed direction of unit number 1. In reality, the angle of a rotation factor is relative to its target position number's direction.

The position stretching and position rotation are two basic motion types in the physical world. From the perspective of describing motion of points in a plane, numbers may be classified into two types: real numbers for position stretching, and rotation factors for position rotation. Multiplying a real number to a position number or position vector stretches the target's magnitude without changing the target's direction or orientation. On the other hand, multiplying a rotation factor to a position number or position vector rotates the target without changing the target's magnitude. Nature turns out to be magically perfect. The existences of real number set $\mathbb{R}$ and the rotation factor set $\mathbb{E}$ complement each other and complete the representation for the two basic motion types.

7. Understanding and utilizing rotation factors

7.1. Rotation generators and rotation targets

Rotation factors are inherently related to the concept of motion. Principally, the set of rotation factors is derived from Equation (16). The left side of the equation indicates the motion represented by the position vector change via rotation. The right side of the equation contains the orthogonal rotation factor i that causes the rotation with the direction of the position vector's instant change being perpendicular to the position vector. It is the concept of motion that brings the orthogonal rotation factor into being. Rotation factors cause rotations, and may be considered as rotation generators.

It is important to note that a rotation factor itself will not rotate and needs a target of vector or number for the rotation generation. For example, in Figure 8, the target is real number r, which is rotated to r$e^{i\theta}$ by rotation factor $e^{i\theta}$. And in Figure 9, the target is unit 1 (denoted by vector), which is rotated to $1 \cdot i$ by orthogonal rotation factor $i = e^{i\frac{\pi}{2}}$.

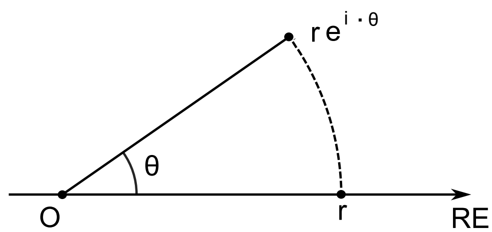

Figure 8. A real number as rotation target of a rotation factor of angle θ

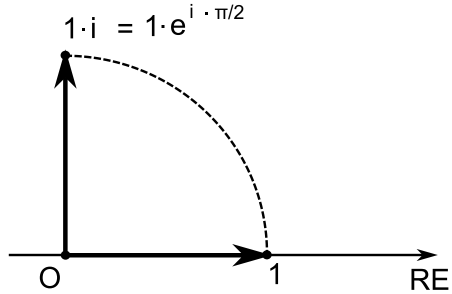

Figure 9. Unit 1 vector as a rotation target of an orthogonal rotation factor i

7.2. Understanding basic mathematical constants in rotation factors

In a rotation factor $e^{i\theta}$, there appear directly or indirectly the three important mathematical constants: Euler's number e, orthogonal rotation factor i, and π. The i appears because the fundamental equation of position number rotation (22) dictates the direction of the derivative of position number with respect to rotation angle at a point be orthogonal to the implied direction of the point; the Euler's number e appears because the derivative of rotation factor q is q itself in (27); and π appears indirectly because a specific angle θ is often expressed as π or a fraction of π.

Especially, the previous result in (4) with n=2 leads to q(π) $= -1$. That is, the q value of angle π is known and the formula (28) for the angle gives q(π) $= e^{i\pi}$. Then it follows that

$$e^{i\pi} = -1 \tag{31}$$

The meaning behind this beautiful Euler's identity is that multiplying the rotation factor of angle π to a position number or a postion vector in a plane makes the implied direction of the number or vector rotate by 180° with the resultant direction being reversed.

7.3. The rotation factor set giving rise to 2D arithmetic

Discovering the existence of the rotation factor set and understanding its properties and implications give rise to two-dimensional (2D) arithmetic.

The position number set $\mathbb{P}$ represents all points in a 2D plane, while the real number set $\mathbb{R}$ represents all points in a 1D line. The rotation factor set $\mathbb{E}$ covers all angles for rotation with the orthogonal rotation factor i being one of its members for the angle $\frac{\pi}{2}$. Anyone rotation factor in the $\mathbb{E}$ set has a magic property of making any position number (which includes any real number) to rotate to its angle relative to the direction of the position number by multiplication. The 2D plane is an abstract and linkage to the physical world. The existence of the rotation factor set with the magic property is a gift from Nature.

One of the main benefits of 2D arithmetic is to allow elementary students use the familiar position and rotation concepts of the physical world, and solve immediate mathematics problems from a higher dimension perspective than 1D without the deeper knowledge behind the rotation factors.

Next, how 2D arithmetic may be readily constructed for such a purpose is illustrated. Denote a rotation factor of angle θ as $[\theta] = e^{i\theta}$ to hide the complexity to a beginner. The square brackets [] appear to be a good selection in terms of being less conflicting with existing conventions, common and direct for typing from a standards computer keyboard. Rotation factors may also be called rotation numbers in direct comparison with real numbers.

In Figure 10, RE denotes a real number axis and OR denotes the axis orthogonal to RE. Point O is the origin. The degree unit is used for angles in this illustration. Multiplying rotation number [30] to the number 3 makes the point rotate by 30 ° relative to the number's direction and reach 3[30]; multiplying the position number 3[30] by 2 makes the point stretch to 3[30]2; and multiplying rotation number [15] to 3[30]2 makes the point rotate to 3[30]2[15], which is equal to 6[45]. It is noted that [30][15] = [30 + 15] = [45], due to the fact that multiplying two exponential functions results in the exponents being added. A mobile phone app calculator may be developed and used to calculate the rotation numbers.

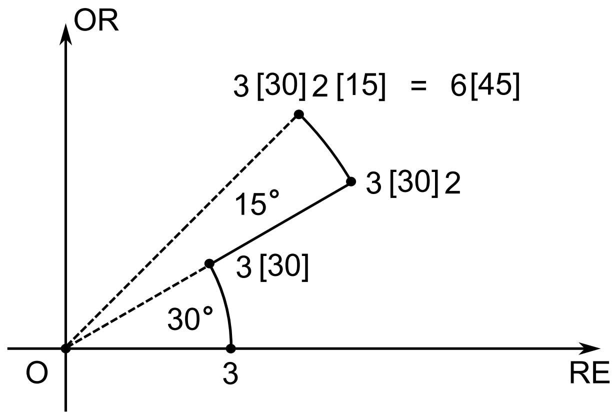

Figure 10. Illustration of 2D arithmetic

7.4. Applying rotation factors to vectors in a plane

In addition to rotating points in a plane, rotation factors can also rotate lines in the plane. In general, there is a direction involved for a rotation. For a point in the plane, its direction is from the origin to the point itself, and the direction may not need to be explicitly specified as there is no ambiguity. But for a line formed by two points in the plane, its direction may need to be denoted by an arrow to identify the direction. Thus, rotating a line with a direction also means rotating a vector.

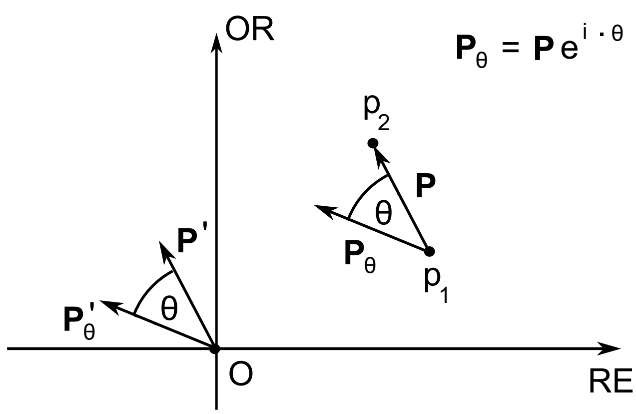

Figure 11. Vector rotated by a rotation factor of angle θ

Figure 11 illustrates the rotation of a vector by a rotation factor of angle θ. In the figure, RE denotes a real number axis, and OR denotes the axis orthogonal to RE. O is the origin point. The point represented by position number p_1 and the point represented by position number p_2 forms a line. The line's direction is from the starting point p_1 to the ending point p_2, and the line with the direction is denoted by vector $\mathbf{P}$.

With respect to the rotation for vector $\mathbf{P}$ by a rotation factor, it is important to note that by the rotation factor definition (1), the rotation is now relative to the vector line's starting point p_1, not the origin O. Thus, applying the rotation factor of angle θ to the vector $\mathbf{P}$ makes the vector rotate counter-clockwise by angle θ relative to the starting point p_1, and the resultant vector is $\mathbf{P}_\theta$ with the same magnitude. That is, $\mathbf{P}_\theta = \mathbf{P}e^{i\theta}$. This again shows the powerfulness of the relativeness property of a rotation factor for rotation operation.

From another viewpoint for conducting the rotation, a vector may be moved parallel to itself without being changed. Therefore, $\mathbf{P}$ may be moved parallelly to $\mathbf{P}'$, and rotated by the rotation factor of angle θ around the origin. The resultant vector is $\mathbf{P}'_\theta$, which is the same as $\mathbf{P}_\theta$, with the identical direction and magnitude.

8. The causality origin of the orthogonal rotation factor i

Next, the origin of the orthogonal rotation factor i is discussed. The i is defined in (5) as $\mathrm{q}(\frac{\pi}{2})$, the rotation factor of angle $\frac{\pi}{2}$. The i is used to derive the rotation factor formula (28) for obtaining rotation factors of all angles. In fact, one may also use a rotation factor of a different angle and obtain the formula. But the result is less succinct. For example, one may use a rotation factor of angle $\frac{\pi}{4}$ and denote it as j. That is, $j = \mathrm{q}(\frac{\pi}{4})$. With j, the derived formula becomes $\mathrm{q}(\theta) = e^{j^2\theta}$, which is less perfect with the extra complexity in comparison with using i. The i is expressed by j as $i = j \cdot j = j^2$. Although a member in the rotation factor set may be used to represent the other members in a formula, they are not the cause to each other from the causality viewpoint.

Up to now, the i is still a symbol as $\mathrm{q}(\frac{\pi}{2})$. One may still ask what is $\mathrm{q}(\frac{\pi}{2})$? To answer this question, it is noticed that $\mathrm{q}(\frac{\pi}{2})$ is a constant that is denoted by i. And π is also a constant. From the causality viewpoint, investigating the cause of the π may provide a hint for the cause of the i.

Since the ratio of a circle's circumference C to its diameter d is a constant, this leads to the definition of π. That is

$$\pi = \frac{\mathrm{C}}{\mathrm{d}} \tag{32}$$

where C and d are variables, but their ratio π is a constant.

By the same token, from (16), the constant i may be defined as

$$i = \frac{\frac{\partial \mathbf{P}}{\partial \theta}}{\mathbf{P}} \tag{33}$$

where the derivative of position vector with respective to rotation angle $\frac{\partial \mathbf{P}}{\partial \theta}$ and the position vector $\mathbf{P}$ are equivalent to variables, but their ratio is a constant. It is the existence of the ratio being constant that causes the existence of the i as well as the existence of the rotation factor set $\mathbb{E}$. It is important to point out that the i is a special non-real number constant and represents an orthogonal rotation as the direction of the derivative term $\frac{\partial \mathbf{P}}{\partial \theta}$ is perpendicular to that of the vector term $\mathbf{P}$.

The existence of the constant i reflects the property of Nature in the physical world for rotations and related periodic motions such as waves, and it is no wonder that the i also appears in physics such as in Schrödinger equation for quantum mechanics [6].

9. Constructing Cartesian coordinate system with rotation factors and deriving Euler's formula

Now, there exists the rotation factor set $\mathbb{E}$ besides the real number set $\mathbb{R}$. Next, a Cartesian coordinate system is constructed with the two sets by applying rotation factors to real numbers.

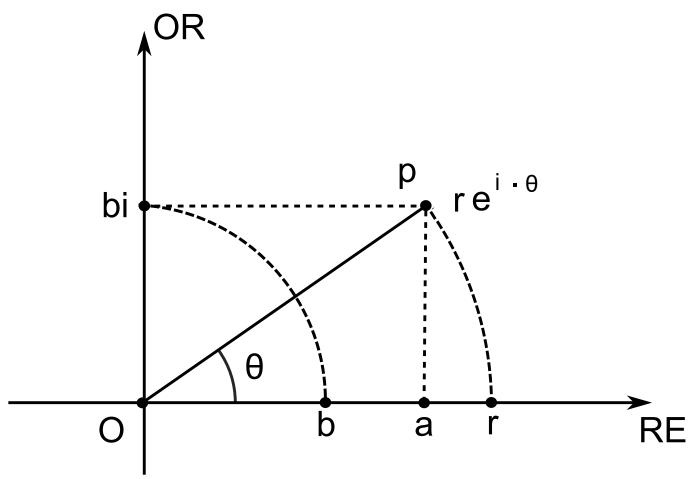

Figure 12. Cartesian coordinate system formed from real numbers and rotation factors

In Figure 12, RE denotes a real number axis. OR denotes the axis orthogonal to RE. Point O is the origin. The point represented by a real number r in the RE axis has the direction of the positive RE axis. Applying a rotation factor of an arbitrary angle θ to number r makes the point rotate by angle θ to the point represented by position number p or $\mathrm{r}e^{i\cdot\theta}$. That is

$$\mathrm{p} = \mathrm{r}e^{i\cdot\theta} \tag{34}$$

The point represented by number a in the RE axis, and the point represented by p, form a line that is parallel to the OR axis. The distance between the origin and the point represented by number b is the same as the distance between the point represented by p and the point represented by a. Applying the orthogonal rotation factor i to number b makes the point rotate to the point bi in the OR axis. By the rotation factor concept, bi has a direction that is perpendicular to the RE axis and points upward. In the Cartesian coordinate system, the position number p is expressed as

$$\mathrm{p} = \mathrm{a} + \mathrm{b}i \tag{35}$$

In Figure 12, from the definition of sine and cosine, $\mathrm{a} = \mathrm{r}\cos(\theta)$ and $\mathrm{b} = \mathrm{r}\sin(\theta)$. Thus, (35) becomes $\mathrm{p} = \mathrm{r}\cos(\theta) + i\mathrm{r}\sin(\theta)$, which, together with (34), leads to

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{36}$$

The result (36) is Euler's formula. The new interpretation of the formula is that multiplying rotation factor $e^{i\theta}$ to the unit number 1 makes the number rotate to a position with the component in the real axis direction being $\cos(\theta)$ and the component in the direction orthogonal to the real axis being $i\sin(\theta)$. The deep meaning behind the formula turns out to be the concept of rotation factors and the existence of the rotation factor set in Nature.

The position number p in (35) is equivalent to the definition of a complex number. All complex numbers in the 2D plane form the complex number set $\mathbb{C}$. The orthogonal rotation factor i here is equivalent to the imaginary unit. The discovery of the rotation factor set along with the orthogonal rotation factor i finally reveals the causality origin of the mysterious imaginary unit and shows the realness of the i with the origin of the ratio in (33) being measurable in the physical world.

10. Summary

10.1. Fundamental equation of position vector rotation

The fundamental equation of position vector rotation has been discovered. The equation governs the rule for the rotation of a point in a plane.

10.2. Causality origin of the imaginary unit

The causality origin of the imaginary unit i has been finally revealed. It is the concept of rotation motion that brings the imaginary unit into being. The imaginary unit is equivalent to the orthogonal rotation factor, which emerges in the fundamental equation of position vector rotation. The equation dictates that the derivative of position vector with respective to rotation angle equals the position vector multiplied by the orthogonal rotation factor. The ratio of the derivative to the position vector happens to be a constant, which is also the orthogonal rotation factor itself. Thus, the imaginary unit has been shown to be real and directly involved in the reality through the position rotation.

10.3. Existence of the rotation factor set

Based on the fundamental equation of position number rotation, the existence of the rotation factor set has been found and the formula for the set has been obtained. The rotation factor set covers all angles for rotation with the orthogonal rotation factor i being one of its members. Anyone rotation factor in the set has a magic property of making any position point to rotate to its angle relative to the direction of the position point by multiplication. Position stretching and position rotation are two basic motion types in the physical world. Nature is inherently perfect and complete. Through multiplication, the rotation factor set and real number set complement each other, with the former for position rotation and the latter for position stretching.

10.4. The 2D arithmetic

The existence of the rotation factor set and the understanding of its properties immediately give rise to the 2D arithmetic with simplicity and familiarity. Based on the readily known position concept, introduction of the 2D arithmetic to elementary education would bring in the higher dimension perspective and empowerment.

10.5. Constructing Cartesian coordinate system and deriving Euler's formula based on rotation factors

Rotation factors may be used to construct a Cartesian coordinate system. The result is equivalent to the complex number system with the imaginary unit being the orthogonal rotation factor and the Euler's formula being the consequence of the rotation factor set.

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